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CHAOTIC ASPECTS OF THE SHIFT MAP ON THE BI-SIDED FULL *m*-SHIFT

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ABSTRACT:

The aim of this paper is to study some dynamical aspects of the shift map σ on the *bi-sided full m-shift* $X_{[m]} = \Sigma_m$. We mainly prove that it is *Devaney chaotic* (*DevC*), *Auslander-Yorke chaotic* and *generically* δ*-chaotic*. We also establish that σ has *chaotic* as well as *modified weakly chaotic dependence on initial conditions*. Further we have derived the *zeta function* for this map and calculated the *entropy* for the *full m-shift*.

KEYWORDS: Shift Space, Shift Map, Topological Transitivity, Topological Mixing, Sensitive Dependence, Zeta Function, Entropy.

2010 Mathematics Subject Classifications: 37B10, 37D45, 37C15

1. INTRODUCTION:

Shifts, particularly *shifts of finite type* or *Markov shifts* [1], as dynamical systems, have some additional advantages over other general dynamical systems. It is seen that a Markov shift has very close links with graphs, transition matrix and linear algebra and also with the probability matrix [1]. Another important aspect is that the study of shift dynamical systems facilitates us in two ways: (i) it gives proper knowledge about their individual dynamics and (ii) it provides good information about the dynamical systems represented by them or topologically conjugate [1, 2] to them. Finite type shifts are sub-shift spaces of the full shifts and hence it becomes an ardent need to know the dynamical nature of these full shifts to be able to analyse the dynamical properties of other Markov shifts. For this reason we first give below a description of full shifts, discuss the notions related to them as well as to other general shifts and mention some important facts which will be useful in our future implementations. In this paper we have established some dynamical aspects of the shift map σ [3, 12] on the full *m*-shift. *Devaney chaos* (*DevC*) [4, 5, 6], *Auslander-Yorke chaos* [8] and *generic* δ*-chaos* [8, 9] of this map have been proved here. *Chaotic* as well as *modified weakly chaotic dependence on initial conditions* [9] have also been established for this map. In addition to these, we have derived the *zeta function* [1] of this *Markov chain* [1] and calculated the entropy of the full *m*-shift.

2. **PRELIMINARY DISCUSSIONS AND BASIC RESULTS:**

***Definition* 2.1: *Li-Yorke Pairs* [8, 10, 11]:** A pair $(y,z)\in X^2$ in a topological dynamical system (X,f) is called a *Li-Yorke* pair with modulus $\delta>0$ if we have (i) $\lim_{n\to\infty} Sup\, d(f^n(y), f^n(z)) \geq \delta$ and also (ii) $\lim_{n\to\infty} Inf\, d(f^n(y), f^n(z)) = 0$.

***Definition* 2.2: *Weakly and modified weakly chaotic dependence on initial conditions* [9]:** A dynamical system (X,f) is called weakly (resp. modified weakly) chaotic dependence on initial conditions if for any $x\in X$ and every neighbourhood $N(x)$ of x, there are points $y,z\in X$ [$y\neq x, z\neq x$ in modified weakly case] such that $(y,z)\in X^2$ is a *Li-Yorke* pair.

***Definition* 2.3: *Generically δ-Chaotic maps* [8, 9]:** A continuous map $f: X\to X$ on a compact metric space X is *generically δ-chaotic* if $LY(f,\delta)$, the set of all the Li-Yorke pairs in X, is residual in X^2.

Proposition 2.1[4, 5, 8]: ***A topological dynamical system $f: X\to X$ is topologically transitive if for every pair of non-empty open sets U and V of X, there exists a positive integer $n\in \mathrm{N}$ such that $f^n(U)\bigcap V\neq\phi$.***

Proposition 2.2 [8]: ***If $f: X\to X$ is a continuous topologically mixing map on a compact metric space X, then f is also topologically weak mixing.***

Proposition 2.3 [8]: ***If a continuous map $f: X\to X$ on a compact metric space X is topologically weak mixing, then it is generically δ-chaotic on X with $\delta = diam(X)$.***

2.1: Full Shifts, Shift Spaces and Shifts of finite Type

The set $\mathcal{A}^{\mathrm{Z}}$ of all two-sided sequences of symbols, also called *letters*, from a finite set $\mathcal{A}$, called the *alphabet*, is the *bi-sided full $\mathcal{A}$-shift* [1, 7]or simply the *full $\mathcal{A}$-shift*. Generally $\mathcal{A}$ contains typical symbols like 0, 1, 2, 3…or a, b, c, d... etc. The full shift over the alphabet $\mathcal{A}$= {0, 1, 2, …, *m*-1}is termed as the *full m-shift* and it is generally denoted by Σ_m *or* $\mathrm{X}_{[m]}$. A typical point *x* in a shift is denoted as

$$x = ...x_{-3}x_{-2}x_{-1}\cdot x_0x_1x_2x_3....... \text{ where } x_i^{'s}\in alphabet$$

A *word* or a *block* of *length k* or simply a *k-block* over $\mathcal{A}$ is a finite sequence of symbols from the alphabet $\mathcal{A}$ of the type $x_{\lambda_1}x_{\lambda_2}x_{\lambda_3}....x_{\lambda_k}$. For $i, j(>i)\in\mathbb{Z}$, $x_{[i,j]}$ denotes the block $x_i x_{i+1} x_{i+2}....x_j$ of coordinates of the point x from the i-th position to the j-th position. The block $x_{[-k,k]} = x_{-k}x_{-k+1}.....x_k$, $k\in\mathbb{N}$, is generally known as the central *(2k+1)*-block of x and the role of the central blocks of points are very essential in studying the dynamics of the full shifts as well as other shift spaces. If u, v be two blocks of letters over $\mathcal{A}$, then uv represents the block of length $|u|+|v|$ obtained by concatenating the letters in v at the tail of u. If $\mathcal{F}$ is a collection of some blocks over $\mathcal{A}$, then $\mathrm{X}_{\mathcal{F}}$ represents the subset of all the sequences in $\mathcal{A}^{\mathbb{Z}}$ which do not include any block in $\mathcal{F}$. $\mathcal{F}$ in this context is known as the collection of *forbidden blocks*. Shifts or shift spaces [1, 7, 12] X are subsets of a full shift $\mathcal{A}^{\mathbb{Z}}$ such that $X=\mathrm{X}_{\mathcal{F}}$ for some collection $\mathcal{F}$ of forbidden blocks. ***Shifts of finite type*** or ***Markov shifts*** are shift spaces which can be described by a finite collection $\mathcal{F}$ of forbidden blocks. The full shifts and the Golden Mean shift are two examples of shifts of this type. Finite type shifts are called ***M-step*** [1] when they can be described by a collection of forbidden blocks all having length equal to (M+1). $B_n(X)$ denotes the set of all the n-blocks which occur in points in the shift space X, called ***allowed blocks in X***. The collection $B(X)=\bigcup_{n=1}^{\infty}B_n(X)$ of all allowed blocks in X is called the ***language of X***. A shift space is ***irreducible*** [1] if for every pair of blocks $u,v\in B(X)$ there exists a block $w\in B(X)$ such that $uwv\in B(X)$.

2.2: Graphs, Adjacency Matrices and Edge Shifts

The correspondence between a graph and its adjacency matrix is well known. For a definite order of listing, the m vertices of a graph give a unique adjacency matrix $A=[A_{IJ}]_{m\times m}$, a square matrix with non-negative integers such that A_{IJ} is the number of edges from the vertex I to the vertex J. Though a different listing order of the vertices may give rise to a different adjacency matrix B, it is not different at all in the sense that A and B are always similar. More precisely, we always have a permutation matrix P such that $B=PAP^{-1}$. Since similar matrices have the same ***Jordan canonical form*** [13], in a certain sense they can be treated as same. On the other hand, a square matrix of order m with non-negative integer entries gives a graph G with a vertex set of m elements. For different labelling of the vertices give isomorphic graphs having identical properties. If G_A denotes the formation of graph of the square matrix A with non-negative integer entries and A_G denotes the formation of adjacency matrix of the graph G for a certain labelling, then, we have the following important facts:

(i) $A = A(G_A)$ and (ii) $G \cong G(A_G)$ [1].

These are the most useful correspondences between graphs and their adjacency matrices. These correspondences indicate that we can use either the graph G or its adjacency matrix A for the specification of the underlined graph, whichever is more convenient in the context. For a graph G with edge set $\mathcal{E}$ and adjacency matrix A, the ***edge shift*** [1] X_G or $\mathbf{X_A}$ is defined to be the shift space over the alphabet $\mathcal{A}=\mathcal{E}$ such that

$$X_A = X_G = \{e = (e_i)_{i \in Z} : t(e_i) = i(e_{i+1})\},$$

where $t(e_i)$ is the terminal vertex of the edge e_i and $i(e_{i+1})$ is the initial vertex of the edge e_{i+1}. This is the connection we have between graphs and shifts. The following propositions reveal this connection more formally and explicitly.

Proposition: 2.4 [1]: *If G is a graph with adjacency matrix A, then the associated edge shift $X_G = X_A$ is a 1-step shift of finite type.*

Proposition: 2.5 [1]: *If G is a graph, then there is a unique sub-graph H of G such that H is essential and $X_G = X_H$.*

Proposition: 2.6 [1]: *Let G be a graph with adjacency matrix A and $m \geq 0$. Then,*

(i) The number of paths of length m from I to J is $[A^m]_{IJ}$, the (I, J)-th entry of A^m.

(ii) The number of cycles of length m in G is $tr(A^m)$, the trace of A^m and this equals the number of points in X_G with period m.

Proposition 2.4 implies that every graph correspond a shift of finite type while Proposition 2.5 expresses that not the whole graph is essential to construct the associated shift, only the largest essential sub-graph of the given graph is sufficient for this purpose. The implications of the Proposition 2.6 will be more useful to derive the number of periodic points of a shift space if we can represent it by a graph.

Shifts of finite type attract more our attentions due to their simplest representation using a finite directed graph and hence questions about the shifts become the questions about their graphs or equivalently the questions about the graphs' adjacency matrices which can be answered more easily and perfectly by using basic results from linear algebra.

2.3: Graph Representation of Shifts and Vertex Shifts

Every shift of finite type is not an edge shift [1]. Golden mean shift is a good example in support of this fact. But any shift of finite type can be recoded, using the higher block presentation, to become an edge shift. In fact, for an *m*-step shift *X* of finite type, there is a graph *G* such that $X^{[m+1]} = X_G$. Here $X^{[m+1]}$ is the image of the shift space *X* under the (*m*+1)th higher block code $\beta_{m+1} : X \to (A_X^{[m+1]})^Z$ given by $(\beta_{m+1}(x))_i = x_{[i,i+m]}, x \in X$, where $(A_X^{[m+1]})^Z$ is the full shift over the alphabet $A_X^{[m+1]} = B_{m+1}(X)$, the collection of all the allowed (*m*+1)-blocks in *X*.

Using a transition matrix, a non-negative matrix with entries either 0 or 1, one can obtain a shift of finite type. The shift obtained in this way is known as a *vertex shift* [1]. The formal definition of such shifts has been given below:

Let *B* be a transition matrix of order $m \times m$. Then it is the adjacency matrix of a graph *G* such that between any two vertices there exists at most one edge. The vertex shift of *B* is a shift space denoted by $\hat{X}_B = \hat{X}_G$ and is defined by

$$\hat{X}_B = \hat{X}_G = \{x = (x_i)_{i \in Z} \in A^Z : B_{x_i x_{i+1}} = 1, \forall i \in Z, A = \{1,2,3,4,......., m\}\}$$

Vertex shifts are1-step shifts of finite type. For, $\hat{X}_B = X_F$ where $F = \{ij : B_{ij} = 0, i, j \in A\}$. These shifts are also called *topological Markov chains* [1, 14]. The topological Markov chain corresponding to the transition matrix *B* is also denoted by Σ_B. We have the following proposition that shows the relations among edge shifts, vertex shifts and shifts of finite type:

Proposition: 2.7[1]:

- *(i)* *Up to a renaming of symbols, every 1-step shift of finite type is same as the vertex shift.*
- *(ii)* *Up to a renaming of the symbols, every edge shift is a vertex shift.*
- *(iii)* *If X is a 1-step shift of finite type, then $X^{[m]}$ is a 1-step shift of finite type, equivalently a vertex shift. In fact, there is a graph G such that $X^{[m]} = \hat{X}_G$ and $X^{[m+1]} = X_G$.*

Though vertex shift is very simple to describe, edge shifts have been mostly preferred in most of the applications due to economy of expression of its adjacency matrix. Further, in case of vertex shifts, certain operations on matrices do not preserve the property of being 0-1 (transition) matrices.

2.4: The full m-shift as a topological dynamical system (TDS) and cylinder sets

A *topological dynamical system* [TDS] is a pair (X, f) where *X* is a compact metric space

[15] and f is a continuous transformation on X. The full m-shift Σ_m is a TDS under the metric d_ρ and transformation σ on Σ_m defined as below:

For $\rho > 1$ and $x = (x_i)_{i=-\infty}^{\infty}, y = (y_i)_{i=-\infty}^{\infty} \in \Sigma_m$, the mapping $d_\rho : \Sigma_m \times \Sigma_m \to R$ defined by

$$d_\rho(x, y) = \begin{cases} \rho^{-k} \; if \;\; x \neq y \; and \; k \in \mathrm{N} \; is \; greatest \;\; s.t. \; x_{[-k,k]} = y_{[-k,k]} \\ 1 \quad if \;\; x_0 \neq y_0 \\ 0 \quad if \;\; x = y \end{cases}$$

is easily seen to be a metric for Σ_m. From definition it is clear that two points in Σ_m are close to each other if they admit in a large central block. Under this metric Σ_m is a compact metric space [1, 15]. Also, the shift transformation σ on the full m-shift Σ_m defined by $\sigma(x) = ...x_{-2}x_{-1}x_0 \cdot x_1x_2x_3.....$, i.e. σ shifts every letter in x one place to the left, is a continuous map [1, 3] and hence (Σ_m, σ) is a *topological dynamical system* (TDS). The concept of open and closed sets plays a very essential role in the study of metric spaces. In the full shift spaces we have sets, known as *cylinders*, which are both open and closed at the same time. These cylinders, particularly the class of *symmetric cylinders* and *admissible symmetric cylinders* in shift spaces, are very important in the studies of shift spaces as topological dynamical systems. Because these class of cylinders form bases for the shift spaces. Therefore, we need the formal definition of these important terms.

If $l, n \in \mathrm{N}$ and $a_i \in \{0,1,2,......., m-1\}$, $-l \leq i \leq n$, then a ***cylinder*** $C_{-l,n}(a_{-l}, a_{-l+1}, ..., a_n)$ is a subset of Σ_m defined as:

$$C_{-l,n}(a_{-l}, a_{-l+1}, ..., a_n) = \{x = (x_i)_{i=-\infty}^{i=\infty} \in \Sigma_m : x_i = a_i, \forall -l \leq i \leq n\}.$$

For $n \in \mathrm{N}$, $C_{-n,n}(a_{-n}, a_{-n+1}, ..., a_n)$ is called a ***symmetric cylinder***. In case of a *topological Markov chain* [9] $\Sigma_B \subset \Sigma_m$ corresponding to a transition matrix B, a cylinder $C_{-l,n}(a_{-l}, ..., a_n)$ is ***an admissible cylinder*** if $B_{a_i a_{i+1}} = 1, \forall -l \leq i < n$ and a cylinder $C_{-n,n}(a_{-n}, ..., a_n)$ is ***an admissible symmetric cylinder*** if $B_{a_i a_{i+1}} = 1, \forall -l \leq i < n$.

The following three well known propositions related to cylinder sets are more important and have been extensively used in the proofs of some main theorems deduced by us.

Proposition: 2.8: *If* $\rho > 2m-1,$ *then any non-empty open set* $U \subset \Sigma_m$ *contains a symmetric cylinder* $C_{-n,n}(a_{-n},.....,a_n)$.

Proposition: 2.9: *If* $\rho > 2m-1,$ *then any non-empty open set* $U \subset \Sigma_B$ *contains an admissible symmetric cylinder* $C_{-n,n}(a_{-n},.....,a_n)$.

Proposition: 2.10: *If* $\rho > 2m-1,$ *then for* $\varepsilon = 1/\rho^n, C_{-n,n}(x_{-n},......,x_n) = B_{d_\rho}(x, 1/\rho^n)$ *where* $x = (x_i)_{i=-\infty}^{i=\infty}$ *contains the central block* $x_{[-n,n]} = x_{-n}.....x_{-1} \cdot x_0......x_n$.

2.18: ***Irreducible and Aperiodic Matrices***:

Irreducibility and aperiodicity of matrices are two more essential concepts in linear algebra as well as in dynamical systems. A transition matrix A is said to be ***irreducible*** if for any $i, j \in \mathrm{N}$, $1 \le i, j \le m, \exists n \in \mathrm{N}$ (possibly dependent on i, j) such that $(A^n)_{ij} > 0.$ i.e. the $(i, j)^{th}$ entry of A^n is positive.

On the other hand, a transition matrix is ***aperiodic*** if there exists $n \in \mathrm{N}$ such that for any $1 \le i, j \le m, (A^n)_{ij} > 0.$ i.e. the matrix A^n is positive. From the definitions of irreducible and aperiodic matrices it is clear that *an aperiodic matrix is always irreducible*.

3. The Main Results:

Proposition: 3.1[1]: *If* $\sigma : \Sigma_A \to \Sigma_A$ *be a topological Markov chain corresponding to the transition matrix A, then,*

(i) *A is irreducible if and only if* $\sigma : \Sigma_A \to \Sigma_A$ *is topologically transitive.*

(ii) *If A is aperiodic, then,* $\sigma : \Sigma_A \to \Sigma_A$ *is topologically mixing.*

Proof: In the proof of this proposition the following Lemma have been extensively used.

Lemma [1]: *If* $A^n > 0$ *for some* $n \in \mathrm{N}$ *, then for any integer* $r > n$ *we also have that* $A^r > 0$.

Proof of the Lemma: The concepts of graph of the transition matrix A have been basically used in the proof of this Lemma.

If $A^n > 0$ for some $n \in \mathrm{N}$, then this means that for every $j,\ 1 \le j \le m$ where m is the order of the matrix A, there exists a positive integer $k_j \in \mathrm{N}$ such that $A_{k_j j} = 1$. For, otherwise, if $A_{kj} = 0$ for all $1 \le k \le m$, then the vertex v_j of the corresponding graph of the matrix A cannot be reached from any other vertex v_k. In this case, there cannot have any path of length k reaching the vertex v_j. This contradicts our assumption that $A^n_{ij} > 0$ $(\because A^n > 0)$.

Now by induction we show that for any $r \geq n$. The result is true for $r = n$ by our assumption. Let it be true for some $r > n$ such that $A^r > 0$ and let $1 \leq i, j \leq m$. Then, by our first remark, for every j, there exists $k_j \in \mathbb{N}$ such that $A_{k_j\, j} = 1$. Further, for all the other $1 \leq k \leq m$, we have $A_{k\,j} \geq 0$. So, we clearly have that

$$A^{r+1}_{ij} = \sum_{r=1}^{m} A^r_{ik} A_{kj} \geq A^r_{ik_j} A_{k_j j} = A^r_{ik_j}.1 = A^r_{ik_j} > 0 \ [\because A^r > 0 \Rightarrow A^r_{ik_j} > 0]$$

This proves that $A^{r+1} > 0$ and hence by induction the Lemma follows.

Proof of the proposition:

(i) Let us first assume that the matrix A is irreducible. We need to show that $\sigma : \Sigma_A \to \Sigma_A$ is topologically transitive. To show this we establish that for non-empty open sets $U, V \subseteq \Sigma_A$, there exists $M \in \mathrm{N}$ such that $\sigma^M(U) \cap V \neq \phi$.

Fix $\rho > 2m - 1$. Then, by proposition 2.8, for the non-empty open sets $U, V \subseteq \Sigma_A$ there exist symmetric cylinders $C_{(-k,k)}(x_{-k}, \ldots, x_k) \subseteq U$ and $C_{(-l,l)}(y_{-l}, \ldots, y_l) \subseteq V$. Now we construct a point $z \in \Sigma_A$ using the central blocks of these symmetric cylinders.

Take $i = x_k$ and $j = y_{-l}$. Then by irreducibility of A, there exists $n \in N$ such that $A^n_{ij} > 0$. This implies that there is a path of length n in G_A that connects the vertex v_{x_k} to the vertex $v_{y_{-l}}$. Let the digits describing this path be $z_0, z_1, \ldots, z_{n-1}, z_n$ where $z_0 = x_k, z_n = y_{-l}$. Clearly, for every non-negative integer i with $0 \leq i \leq n-1$, we have $A_{y_i y_{i+1}} = 1$.

Now, consider the point $z \in \Sigma_A$ such that

$$z = \ldots\ldots x_{-k} \ldots\ldots x_{-2} x_{-1} \cdot x_0 x_1 \ldots\ldots x_k z_1 z_2 \ldots\ldots z_{n-1} y_{-l} \ldots\ldots y_l \ldots\ldots$$

Here, as z contains the central block $x_{-k}, \ldots, x_k$; so, $z \in C_{(-k,k)}(x_{-k}, \ldots, x_k) \subseteq U$. Further, if we take $M = k + n + l$, $\sigma^M(z) = \ldots\ldots y_{-l} \ldots\ldots \underbrace{y_0}_{i=0} \ldots\ldots y_l \ldots\ldots$ and so $\sigma^M(z) \in C_{(-l,l)}(y_{-l}, \ldots, y_l) \subseteq V$. So, it follows that $z \in U \cap \sigma^{-M}(V) \Leftrightarrow \sigma^M(U) \cap V \neq \phi$. That is, the shift map $\sigma : \Sigma_A \to \Sigma_A$ is topologically transitive.

Conversely, let $\sigma:\Sigma_A \to \Sigma_A$ be topologically transitive. We now show that A is irreducible. Let $1 \le i, j \le m$ and take the cylinders $C_0(i) = \{x \in \Sigma_A : x_0 = i\}$ and $C_0(j) = \{y \in \Sigma_A : y_0 = j\}$. Since, cylinder sets are always open as well as closed, so, we can take $C_0(i)$ and $C_0(j)$ as open sets. Let us denote them as U and V respectively. Then by transitivity of $\sigma:\Sigma_A \to \Sigma_A$, there exists $n \in N$ such that $\sigma^n(U) \cap V \neq \phi$.

Now, $\sigma^n(U) \cap V \neq \phi \Leftrightarrow U \cap \sigma^{-n}(V) \neq \phi \Leftrightarrow \exists\ z \in U \cap \sigma^{-n}(V)$

$$\Leftrightarrow \exists z\ s.t.\ z \in U = C_0(i)\ and\ z \in \sigma^{-n}(V = C_0(j))$$

$$\Leftrightarrow \exists z\ s.t.\ z_0 = i\ and\ z_n = j$$

Thus we have got an element $z \in U \subseteq \Sigma_A$ that describes a bi-infinite path on the graph G_A of A such that $z_0 = i$, $z_n = j$ and this gives a path of length n connecting the vertex v_i to v_j. From this it follows that for all i, j with $1 \le i, j \le m$, there exists $n \in N$ such that $A^n_{ij} > 0$. Hence A is irreducible.

(ii) Let A be aperiodic. Then, there exists $n \in N$ such that $A^n > 0$. We show that $\sigma:\Sigma_A \to \Sigma_A$ is topologically mixing. i.e., for any pair of non-empty open sets $U, V \subseteq \Sigma_A$, there exists $M_0 \in \mathrm{N}$ such that $\sigma^M(U) \cap V \neq \phi$ for all $M \ge M_0$. In a similar reasoning as in part (i), both $U, V \subseteq \Sigma_A$ contains symmetric cylinders $C_{(-k,k)}(x_{-k},....,x_k) \subseteq U$ and $C_{(-l,l)}(y_{-l},....,y_l) \subseteq V$. Let $M_0 = n + k + l$. If $M \ge M_0$, then $M = m + k + l$ with $m \ge n$. Also, by the above Lemma, $A^m > 0$ and so $A^m_{x_k y_{-l}} > 0$. Therefore, there exists a path of length m from the vertex x_k to the vertex y_{-l}. So, as in part (i), we can construct a point z of the form

$$z = ...x_{-k}.....x_{-2}x_{-1} \cdot x_0x_1.....x_kz_1z_2......z_{n-1}y_{-l}.......y_l........ \text{ in } \Sigma_A \text{ such that } z \in U \cap \sigma^{-M}(V)$$

and from this it immediately follows that $\sigma^M(U) \cap V \neq \phi$. This is true for any $M \ge M_0$. Hence $\sigma:\Sigma_A \to \Sigma_A$ is topologically mixing.■

Theorem: 3.2: *The shift map* $\sigma:\Sigma_m \to \Sigma_m$ *is topologically transitive as well as mixing.*

Proof :(i) *Topological transitivity of* σ :

Let U and V be any two non-empty open sets in Σ_m. We show that for these two non-empty open sets U and V, there exists a positive integer n such that $\sigma^n(U) \cap V \neq \phi$.

Since *U* and *V* are non-empty, so, we have $x=(x_i)_{i=-\infty}^{\infty}\in U$ and $y=(y_i)_{i=-\infty}^{\infty}\in V$. Again, since *U* and *V* are open sets in Σ_m, so there are open balls $B_{d_\rho}(x,r_1)$ and $B_{d_\rho}(y,r_2)$ such that $B_{d_\rho}(x,r_1)\subseteq U$ and $B_{d_\rho}(y,r_2)\subseteq V$.

Fix $\rho>2m-1$. Now for $r_1,r_2>0$, we can find a positive integer *n* such that $\rho^{-n}\le\min\{r_1,r_2\}$. Then clearly $B_{d_\rho}(x,\rho^{-n})\subseteq U$ and $B_{d_\rho}(y,\rho^{-n})\subseteq V$. Also, we have that $B_{d_\rho}(x,\rho^{-n})=C_{-n,n}(x_{-n},....,x_n)$ and $B_{d_\rho}(y,\rho^{-n})=C_{-n,n}(y_{-n},....,y_n)$. Therefore, all the points in $B_{d_\rho}(x,\rho^{-n})$ must agree with *x* in the $(2n+1)$-central block and all the points in $B_{d_\rho}(y,\rho^{-n})$ must agree with *y* in the $(2n+1)$-central block.

We now consider a very typical point $z=(z_i)_{i=-\infty}^{\infty}\in\Sigma_m$ such that $z_i=x_i,\forall i=-n,....,n$ and $z_{n+i}=y_{i-1-n},\forall i=1,2,...,2n+1$. Then clearly the point *z* agrees with *x* in $(2n+1)$-central block and hence $z\in C_{-n,n}(x_{-n},....,x_n)=B_{d_\rho}(x,\rho^{-n})$. Further $\sigma^{2n+1}(z)$ agrees with *y* in $(2n+1)$-central block and so $\sigma^{2n+1}(z)\in C_{-n,n}(y_{-n},...,y_n)=B_{d_\rho}(y,\rho^{-n})$.

Thus $z\in B_{d_\rho}(x,\rho^{-n})\subseteq U$, $\sigma^{2n+1}(z)\in B_{d_\rho}(y,\rho^{-n})\subseteq V\Rightarrow\sigma^{2n+1}(z)\in\sigma^{2n+1}(U),\sigma^{2n+1}(z)\in V$

$$\Rightarrow\sigma^{2n+1}(z)\in\sigma^{2n+1}(U)\bigcap V$$

$$\Rightarrow\sigma^{2n+1}(U)\bigcap V\neq\phi$$

Hence the self-map $\sigma:\Sigma_m\to\Sigma_m$ is topologically transitive. ■

(ii) ***σ is topologically mixing*** : Let *U* and *V* be any two non-empty open sets in Σ_m. Here we need to prove that for the non-empty open sets *U* and *V*, there exists a positive integer $n_0\in\mathrm{N}$ such that $\sigma^n(U)\bigcap V\neq\phi,\forall n\ge n_0,n\in\mathrm{N}$. *U* and *V* being non-empty, we get $x=(x_i)_{i=-\infty}^{\infty}\in U$ and $y=(y_i)_{i=-\infty}^{\infty}\in V$. Again, *U* and *V* being open, there are open balls $B_{d_\rho}(x,r_1)$ and $B_{d_\rho}(y,r_2)$ such that $B_{d_\rho}(x,r_1)\subseteq U$ and $B_{d_\rho}(y,r_2)\subseteq V$. Now for $r_1,r_2>0$, we can find $k\in\mathrm{N}$ such that $\rho^{-k}\le\min\{r_1,r_2\}$. Then clearly $B_{d_\rho}(x,\rho^{-k})\subseteq U$ and $B_{d_\rho}(y,\rho^{-k})\subseteq V$.

Also, for a fixed $\rho>2m-1$, by proposition 2.10 we have that $B_{d_\rho}(x,\rho^{-k})=C_{-k,k}(x_{-k},....,x_k)$ and $B_{d_\rho}(y,\rho^{-k})=C_{-k,k}(y_{-k},....,y_k)$. In this case, every point in $B_{d_\rho}(x,\rho^{-k})$ must agree with *x* in the

$(2k+1)$-central block and every point in $B_{d_\rho}(y,\rho^{-k})$ must agree with y in the $(2k+1)$-central block. We now construct a sequence $\{z_n\}$ points in Σ_m with the help of x, y and k as follows:

$$z_1 = ...x_{-k}......x_{-1}\cdot x_0......x_k y_{-k}........y_k y_{k+1}.........$$

$$z_2 = ...x_{-k}.....x_{-1}\cdot x_0......x_k a_1 y_{-k}.......y_k y_{k+1}........$$

$$z_3 = ...x_{-k}.....x_{-1}\cdot x_0......x_k a_1 a_2 y_{-k}.......y_k y_{k+1}........$$

$$........\ ....\ ....\ ....\ ....\ ....\ ....$$

$$z_i = ...x_{-k}.....x_{-1}\cdot x_0.....x_k a_1 a_2.....a_{i-1} y_{-k}.....y_k y_{k+1}......, i\geq 2, a_i^{'s}\in\{0,1,...,m-1\}$$

Here, every $z_i, i\geq 2$, is constructed by concatenating the words $x_{[-\infty,k]}$, $a_{[1,i-1]}$ and $y_{[-k,\infty]}$, where $a_{[1,i-1]}$ is the word of a fixed sequence $a=(a_i)_{-\infty}^{\infty}$ chosen arbitrarily. Also, we note here that every $z_i, i\geq 1$, agrees with the point x at least in the $(2k+1)$-central block. Therefore, for every $z_i, i\geq 1$, we have $z_i \in C_{-k,k}(x_{-k},....,x_k) = B_{d_\rho}(x,\rho^{-k}) \subseteq U$.

Now, $\sigma^{2k+1}(z_1) = ...x_{-k}......x_k y_{-k}......y_{-1}\cdot \underbrace{y_0}_{i=0}......y_k......\in V$ and $\sigma^{2k+1}(z_1)\in \sigma^{2k+1}(U)$

$$\Rightarrow \sigma^{2k+1}(z_1)\in \sigma^{2k+1}(U)\bigcap V$$

$$\Rightarrow \sigma^{2k+1}(U)\bigcap V \neq \phi.$$

Also, $\sigma^{2k+i-1}(z_i)\in U, \sigma^{2k+i-1}(z_i) = ...x_{-k}....x_k a_1 a_2....a_{i-1} y_{-k}....y_{-1}\cdot \underbrace{y_0}_{i=0}....y_k....\in V, \forall i\geq 2, i\in \mathrm{N}$.

So, $\sigma^{2k+i-1}(U)\bigcap V\neq\phi$, for all $i\geq 2$. Thus, $\sigma^n(U)\bigcap V\neq\phi$, for all $n=2k+i-1\geq n_0=2k+1$.

Hence, the shift map $\sigma:\Sigma_m\to\Sigma_m$ is topologically mixing.■

Remarks: We can alternatively give the proof of this theorem as an immediate consequence of the proposition3.1 which uses irreducibility and aperiodicity. The implementations are as follows:

Consider the matrix $A=[A_{ij}]_{m\times m}$ where $A_{ij}=1$ for all $i,j\in \mathrm{N}$ with $1\leq i,j\leq m$. This transition matrix clearly induces the graph G with m vertices such that there is exactly one and only one edge from every vertex v_i to the vertex v_j. So, in this case clearly we have that $\Sigma_A=\Sigma_m$. Also, $A=[A_{ij}]_{m\times m}$ being positive is aperiodic and hence it is irreducible. Therefore, by Proposition 3.1, $\sigma:\Sigma_m\to\Sigma_m$ is topologically transitive as well as topologically mixing.■

Theorem: 3.3: *The set* $P(\sigma)$ *of all the periodic points of* $\sigma : \Sigma_m \to \Sigma_m$ *is dense in* Σ_m.

Proof: Consider an arbitrary point $x = (x_i)_{i=-\infty}^{\infty} = ...x_{-k}......x_{-2}x_{-1} \cdot x_0 x_1.......x_k....... \in \Sigma_m$. We need to show that for any $\varepsilon > 0$, however small, there is a periodic point $p \in P(\sigma)$ such that $d_\rho(x, p) < \varepsilon$. That is, for an arbitrarily chosen small $\varepsilon > 0$, the ε-neighbourhood of x contains points of $P(\sigma)$. We note that for fixed $\varepsilon > 0$ and $\rho > 1$, we can always find $n \in N$ such that $\rho^{-n} < \varepsilon$. Now, for the arbitrary point $x \in \Sigma_m$, we find a periodic point $p \in P(\sigma)$ satisfying all our requirements mentioned above. Take the point $p \in \Sigma_m$ such that

$$p = ...x_{-n}.....x_0....x_n x_{-n}......x_{-2}x_{-1} \cdot x_0 x_1.......x_n x_{-n}....x_0......x_n...... \in \Sigma_m$$

That is, the point *p* is constructed by concatenating the fixed $(2n+1)$-block $x_{[-n,n]}$ of the given point infinitely in both directions. This can always be done for any arbitrary point *x*.

Since, *x* and *p* agree at least in the $(2n+1)$ central block, so by definition of the metric d_ρ, we have $d_\rho(x, p) \le \rho^{-n} < \varepsilon$. Also, $p \in \Sigma_m$ thus constructed is clearly a periodic point and hence $p \in P(\sigma)$. Thus for every point $x \in \Sigma_m$, we always have a point $p \in P(\sigma)$ which is at a distance less than an arbitrarily chosen small quantity $\varepsilon > 0$. Hence $P(\sigma)$ is dense in Σ_m. ■

Theorem: 3.4: *The shift map* $\sigma : \Sigma_m \to \Sigma_m$ *has sensitive dependence on initial conditions with the sensitivity constant* $\delta = 1$.

Proof: For simplification of the proof, we first fix ρ such that $\rho > 2m - 1$.

Now we show that for any $\varepsilon = 1/\rho^n = \rho^{-n} (n \in \mathrm{N})$ and $x = (x_i)_{i=-\infty}^{i=\infty} \in \Sigma_m$, there always exists a point $y = (y_i)_{i=-\infty}^{\infty} \in \Sigma_m$ in the ε-neighbourhood of *x* such that $x_{n+1} \ne y_{n+1}$. By proposition 2.7, for $\rho > 2m - 1$, $\varepsilon = 1/\rho^n$ we always have that $C_{-n,n}(x_{-n},...., x_n) = B_{d_\rho}(x, 1/\rho^n)$.

Suppose $N_\varepsilon(x)$ denotes the ε-neighbourhood of *x*. Then, clearly this ε-neighbourhood is nothing but the open ball $B_{d_\rho}(x, \varepsilon) = B_{d_\rho}(x, 1/\rho^n)$ .Let, $y \in B_{d_\rho}(x, \varepsilon) = N(x)$ with $x \ne y$ .We claim that it is always possible to have such a point $y \in \Sigma_m$ such that $y \in B_{d_\rho}(x, \varepsilon)$. For, if we take $y = (y_i)_{i=-\infty}^{i=\infty} \in \Sigma_m$ with $x_{[-n,n]} = y_{[-n,n]}$ and $x_{n+1} \ne y_{n+1}$, then, $y \in C_{-n,n}(x_{-n},...., x_n) = B_{d_\rho}(x, 1/\rho^n) = B_{d_\rho}(x, \varepsilon)$. Now, assume that $\varepsilon > 0$ be arbitrarily small number. In this case we can find a positive integer $n \in \mathrm{N}$ such that

$\rho^{-(n+1)} \le \varepsilon \le \rho^{-n}$. If we take $\varepsilon_1 = \rho^{-(n+1)}$, then by our argument as before, there exists a point $y = (y_i)_{i=-\infty}^{\infty} \in \Sigma_m$ in the ε_1-neighbourhood of x such that $x_{[-n-1,n+1]} = y_{[-n-1,n+1]}$ and $x_{n+2} \neq y_{n+2}$. Actually there are infinite number of such points. Here $d_\rho(x, y) = \varepsilon_1 = \rho^{-(n+1)} < \varepsilon$.

Again, $x = (x_i)_{i=-\infty}^{\infty}, y = (y_i)_{i=-\infty}^{\infty} \in \Sigma_m$ *where* $x_{[-n-1,n+1]} = y_{[-n-1,n+1]}, x_{n+2} \neq y_{n+2}$

$$\Rightarrow \sigma^{n+1}(x) \neq \sigma^{n+1}(y) \text{ where } (\sigma^{n+1}(x))_0 \neq (\sigma^{n+1}(y))_0$$

$$\Rightarrow d_\rho(\sigma^{n+1}(x), \sigma^{n+1}(y)) = 1 (= \delta)$$

Thus there exists $\delta(=1)$ such that for any $x = (x_i)_{i=-\infty}^{\infty} \in \Sigma_m$ and any neighbourhood $N(x)$ of x, there exists $y = (y_i)_{i=-\infty}^{\infty} \in N(x)$ and $k \in \mathrm{N}$ with $d_\rho(\sigma^k(x), \sigma^k(y)) = 1(= \delta)$.

Hence the theorem follows.■

Theorem: 3.5: *The shift map* $\sigma : \Sigma_m \to \Sigma_m$ *Devaney as well as Auslander-Yorke chaotic.*

Proof: In the theorem 3.2 we have seen that σ is topologically transitive, theorem 3.3 shows that the set $P(\sigma)$ of all the periodic points of σ is dense in Σ_m and in theorem 3.4 it has been established that σ has sensitive dependence on initial conditions. So, it follows that $\sigma : \Sigma_m \to \Sigma_m$ is *Devaney chaotic*. Also, a *Devaney chaotic* map is always an *Auslander-Yorke chaotic*. Hence, σ being a *Devaney chaotic* map is also *Auslander-Yorke chaotic.*■

Theorem3.6: *The shift map* $\sigma : \Sigma_m \to \Sigma_m$ *is generically* δ*-chaotic with* $\delta = diam(\Sigma_m) = 1$.

Proof: In Theorem 3.2 we have established that the shift transformation σ on Σ_m is topologically mixing. Also, it is a well-known fact that a continuous topologically mixing map on a compact metric space is topologically weak mixing. So, the shift transformation σ being a continuous topologically mixing map on the compact metric space Σ_m is topologically weak mixing.

Again, since a continuous topologically weak mixing map on a compact metric space X is generically δ-chaotic on X with $\delta = diam(X)$. Therefore, it follows that the shift transformation σ on Σ_m being a continuous topologically weak mixing map on the compact metric space Σ_m is generically δ-chaotic with $\delta = diam(\Sigma_m) = 1$. ■

Theorem: 3.7: *The Topological Dynamical System* (Σ_m, σ) *has modified weakly chaotic dependence on initial conditions.*

Proof: A dynamical system (X, f) has modified weakly chaotic dependence on initial conditions if for any $x \in X$ and every neighbourhood $N(x)$ of x, there are $y, z \in N(x)$ with $y \neq x,\ z \neq x$ such that $(y, z) \in X^2$ is *Li-Yorke*.

Let $\rho > 2m - 1$ be fixed. Also, let $x = (x_i)_{i=-\infty}^{\infty} \in \Sigma_m$ be an arbitrary point and $N(x)$ be any neighbourhood of x. Then there exists an open set (open neighbourhood) U of Σ_m such that $x \in U \subseteq N(x)$.

Now, since $x \in U$ and U is an open set, so, for some large $n \in \mathrm{N}$ there exists an open ball $B(x, \rho^{-n})$ such that $B(x, \rho^{-n}) \subseteq U \subseteq N(x)$. Also, we note here that $B(x, \rho^{-n})$ is nothing but the symmetric cylinder $C_{-n,n}(x_{-n}, \ldots\ldots, x_n)$. We now find $y, z \in N(x)$ with $y \neq x,\ z \neq x$ such that the pair $(y, z) \in \Sigma_m^2$ is *Li-Yorke*. We recall that a pair $(y, z) \in \Sigma_m^2$ is *Li-Yorke* in (Σ_m, σ) with modulus $\delta > 0$ if $\lim_{n\to\infty} Sup\ d_\rho(\sigma^n(y), \sigma^n(z)) \geq \delta,\ \lim_{n\to\infty} Inf\ d_\rho(\sigma^n(y), \sigma^n(z)) = 0$. Before proving these, we first define some typical words $\mathrm{A}(x, 3n)$, $\mathrm{A}(x, 5n)$, $\mathrm{A}(x, 7n)$ etc. for the simplification of our proof. We define these words using the letters in $x = (x_i)_{-\infty}^{\infty} \in \Sigma_m$ as follows and with the help of these words we construct the points $y, z \in \Sigma_m$:

$$\mathrm{A}(x,2n) = x^*_{2n+1} x^*_{2n+2} \ldots\ldots\ldots\ldots x^*_{4n} x_{4n+1} x_{4n+2} \ldots\ldots\ldots\ldots x_{6n},$$

$$\mathrm{A}(x,6n) = x^*_{6n+1} x^*_{6n+2} \ldots\ldots\ldots\ldots x^*_{8n} x_{8n+1} x_{8n+2} \ldots\ldots\ldots\ldots x_{10n},$$

$$\mathrm{A}(x,10n) = (x^*_{10n+1} x^*_{10n+2} \ldots\ldots\ldots\ldots x^*_{12n} x_{12n+1} x_{12n+2} \ldots\ldots\ldots\ldots x_{14n}), \text{ …. and so on.}$$

Note that each of the above words contains $4n$ letters, first $2n$ of which are the m-nary complements of the corresponding letters in x and the rest $2n$ letters are just the letters in the corresponding positions in x. In all the above words the *m*-nary complement x_k^* of the letter x_k is given by $x_k^* = (m-1) - x_k, \forall k,\ x_k$. Now we take

$$y = \ldots\ldots x_{-n} \ldots\ldots x_{-1} \cdot \underbrace{x_0}_{i=0} x_1 \ldots\ldots x_n x^*_{n+1} x^*_{n+2} \ldots\ldots x^*_{2n} x_{2n+1} x_{2n+2} \ldots\ldots x_{6n} x_{6n+1} x_{6n+2} \ldots\ldots\ldots\ldots$$

And $z = ...x_{-n}....x_{-1}\cdot \underbrace{x_0}_{i=0}.....x_n x^*_{n+1}.....x^*_{2n}\mathrm{A}(x,2n)\mathrm{A}(x,6n)\mathrm{A}(x,10n)\mathrm{A}(x,14n).........$

With these notations we now prove the theorem as follows:

Since, y and z agree with x in the $(2n+1)$-central block, so by definition of d_ρ we get, $d_\rho(x,y)=\rho^{-n}$, $d_\rho(x,z)=\rho^{-n}$. Also, since every symmetric cylinder is closed, the ball $B(x,\rho^{-n})$ being a symmetric cylinder is closed and hence $y,z\in B(x,\rho^{-n})\subseteq U\subseteq N(x)$.

Here, we note that z contains infinitely many words of the type $\mathrm{A}(x,2(2k-1)n)$, $k\in \mathrm{N}$, containing $4n$ letters each.

Also, $\sigma^{2n+1}(y) = x_{[-\infty,n]}x^*_{[n+1,2n]}\cdot \underbrace{x_{2n+1}}_{i=0}x_{2n+2}....x_{3n}x_{3n+1}....x_{4n}x_{4n+1}....x_{5n}x_{5n+1}.....x_{6n}x_{6n+1}.....$

$$\sigma^{2n+1}(z) = x_{[-\infty,n]}x^*_{[n+1,2n]}\cdot \underbrace{x^*_{2n+1}}_{i=0}......x^*_{4n}x_{4n+1}......x_{5n}x_{5n+1}......x_{6n}x^*_{6n+1}......x^*_{8n}x_{8n+1}......$$

$$\sigma^{5n+1}(y) = x_{[-\infty,n]}x^*_{[n+1,2n]}x_{2n+1}......x_{3n}x_{3n+1}......x_{4n}x_{4n+1}.......x_{5n}\cdot \underbrace{x_{5n+1}}_{i=0}.......x_{6n}x_{6n+1}......$$

$$\sigma^{5n+1}(z) = x_{[-\infty,n]}x^*_{n+1}.......x^*_{4n}x_{4n+1}x_{4n+2}......x_{5n}\cdot \underbrace{x_{5n+1}}_{i=0}x_{5n+2}......x_{6n-1}x_{6n}x^*_{6n+1}x^*_{6n+2}.......$$

Here $(\sigma^{2n+1}(y))_0 \neq (\sigma^{2n+1}(z))_0$ and $\sigma^{5n+1}(y)$, $\sigma^{5n+1}(z)$ agree in $(2n-1)$-central block.

Therefore, $\underset{n\to\infty}{Lt}\sup_n d_\rho(\sigma^n(y),\sigma^n(z)) \geq \underset{n\to\infty}{Lt}\, d_\rho(\sigma^{2n+1}(y),\sigma^{2n+1}(z)) = \underset{n\to\infty}{Lt}\, 1 = 1$

Again, $0 \leq \underset{n\to\infty}{Lt}\inf_n d_\rho(\sigma^n(y),\sigma^n(z))$

$$\leq \underset{n\to\infty}{Lt}\, d_\rho(\sigma^{5n+1}(y),\sigma^{5n+1}(z)) = \underset{n\to\infty}{Lt}\,\rho^{-(n-1)} = 0$$

Now, $0 \leq \underset{n\to\infty}{Lt}\inf_n d_\rho(\sigma^n(y),\sigma^n(z)) \leq 0 \Rightarrow \underset{n\to\infty}{Lt}\inf_n d_\rho(\sigma^n(y),\sigma^n(z)) = 0$.

Thus $\underset{n\to\infty}{Lt}\sup_n d_\rho(\sigma^n(y),\sigma^n(z)) \geq 1$ and $\underset{n\to\infty}{Lt}\inf_n d_\rho(\sigma^n(y),\sigma^n(z)) = 0$.

Hence, $(y,z)\in \Sigma^2_m$ is a *Li-Yorke pair* with modulus $\delta = 1 > 0$. Consequently, the dynamical system (Σ_m, σ) has modified weakly chaotic dependence on initial conditions.■

Theorem: 3.8: *The dynamical system* (Σ_m,σ) *has chaotic dependence on initial conditions.*

Proof: We first note that a dynamical system (X, f) has chaotic dependence on initial conditions if for any $x \in X$ and every neighbourhood $N(x)$ of x, there is a $y \in N(x)$ such that the pair $(x, y) \in X^2$ is *Li-Yorke*.

Let, $a = (a_i)_{i=-\infty}^{\infty} \in \Sigma_m$ be an arbitrary point and $N(a)$ be any neighbourhood of a. Then there exists an open set (open neighbourhood) U of Σ_m such that $a \in U \subseteq N(a)$. Now, since $a \in U$ and U is open, so there exists an open ball $B_{d_\rho}(a, \rho^{-n})$ for some $n \in \mathrm{N}$ such that $B_{d_\rho}(a, \rho^{-n}) \subseteq U \subseteq N(a)$. Fix $\rho > 2m - 1$ so that $B_{d_\rho}(a, \rho^{-n}) = C_{-n,n}(a_{-n}, ..., a_n)$. Now for our purpose we find a very typical point $b \in B_{d_\rho}(a, \rho^{-n}) \subseteq U \subseteq N(a)$ such that $(a, b) \in \Sigma_m^2$ is *Li-Yorke*.

Using the notations as in Theorem: 3.7 and the letters in $a = (a_1, a_2, a_3, ...a_n, ...) \in \Sigma_m$, we define the words $\mathrm{A}(a,2n)$, $\mathrm{A}(a,6n)$, $\mathrm{A}(a,10n)$,……. etc. as follows:

$$\mathrm{A}(a,2n) = a^*_{2n+1}a^*_{2n+2}......a^*_{4n}x_{4n+1}a_{4n+2}......a_{6n},$$

$$\mathrm{A}(a,6n) = a^*_{6n+1}a^*_{6n+2}......a^*_{8n}a_{8n+1}a_{8n+2}......a_{10n},$$

$$\mathrm{A}(a,10n) = a^*_{10n+1}a^*_{10n+2}......a^*_{12n}a_{12n+1}a_{12n+2}......a_{14n} \text{ and so on.}$$

Here we note that each of the above defined words contains $4n$ letters, first $2n$ of which are the m-nary complements of the corresponding letters in a and the rest $2n$ letters are just the letters in corresponding position of a. In all the above words, the m-nary complement a_k^* of a_k is given by $a_k^* = (m-1) - a_k, \forall k$. Now, using the above words we construct the point b as follows:

$$b = ...a_{-n}....a_{-1} \cdot \underbrace{a_0}_{i=0}.....a_n a^*_{n+1}.....a^*_{2n}\mathrm{A}(a,2n)\mathrm{A}(a,6n)\mathrm{A}(a,10n)\mathrm{A}(a,14n).........$$

From the construction of b it is clear that b agrees with a in $(2n+1)$-central block and so we get $d_\rho(a,b) = \rho^{-n}$. Since $B_{d_\rho}(a, \rho^{-n}) = C_{-n,n}(a_{-n}, ..., a_n)$ is closed and $d_\rho(a,b) = \rho^{-n}$, it follows that $b \in B_{d_\rho}(a, \rho^{-n}) \subseteq U \subseteq N(a)$.

Here, we see that the point b contains infinitely many words containing $4n$ letters each of the type $\mathrm{A}(a,2(2k-1)n)$, $k \in \mathrm{N}$. Also,

$$\sigma^{2n+1}(b) = a_{[-\infty,n]}a^*_{[n+1,2n]} \cdot \underbrace{a^*_{2n+1}}_{i=0}....a^*_{4n}a_{4n+1}....a_{5n}....a_{6n}a^*_{6n+1}....a^*_{8n}a_{8n+1}....$$

And $\sigma^{5n+1}(b) = a_{[-\infty,n]}a^*_{n+1}....a^*_{4n}a_{4n+1}....a_{5n} \cdot \underbrace{a_{5n+1}}_{i=0}a_{5n+2}....a_{6n}a^*_{6n+1}a^*_{6n+2}.....$

Here $(\sigma^{2n+1}(a))_0 \neq (\sigma^{2n+1}(b))_0$ and $\sigma^{5n+1}(a)$, $\sigma^{5n+1}(b)$ agree in $(2n-1)$-central block.

Therefore, $\underset{n\to\infty}{Lt}\sup_n d_\rho(\sigma^n(a),\sigma^n(b)) \geq \underset{n\to\infty}{Lt}\, d_\rho(\sigma^{2n+1}(a),\sigma^{2n+1}(b)) = \underset{n\to\infty}{Lt}\, 1 = 1$

Again, $0 \leq \underset{n\to\infty}{Lt}\inf_n d_\rho(\sigma^n(a),\sigma^n(b))$

$\leq \underset{n\to\infty}{Lt}\, d_\rho(\sigma^{5n+1}(a),\sigma^{5n+1}(b)) = \underset{n\to\infty}{Lt}\, \rho^{-(n-1)} = 0$

Now, $0 \leq \underset{n\to\infty}{Lt}\inf_n d_\rho(\sigma^n(a),\sigma^n(b)) \leq 0 \Rightarrow \underset{n\to\infty}{Lt}\inf_n d_\rho(\sigma^n(a),\sigma^n(b)) = 0$.

Thus $\underset{n\to\infty}{Lt}\sup_n d_\rho(\sigma^n(a),\sigma^n(b)) \geq 1$ and $\underset{n\to\infty}{Lt}\inf_n d_\rho(\sigma^n(a),\sigma^n(b)) = 0$.

Hence, $(a,b) \in \Sigma_m^2$ is a *Li-Yorke pair* with modulus $\delta = 1 > 0$. Consequently, the dynamical system (Σ_m, σ) has chaotic dependence on initial conditions.■

4. *Zeta functions of the shift map* σ:

Let (X, f) be a dynamical system. For $n \in \mathrm{N}$, let $p_n(f)$ be the number of periodic points of period *n*, i.e., $p_n(f) = \left|\{x \in X : f^n(x) = x\}\right|$. Then p_n is a topological invariant [1], i.e. the values of p_n are same for two topologically conjugate dynamical systems. The zeta function of f, denoted by $\zeta_f(t)$, is again a topological invariant that combines all the $p_n^{'s}$. For the dynamical system (X, f) with $p_n(f) < \infty$, $\forall n \in \mathrm{N}$, the zeta function $\zeta_f(t)$ is defined as:

$$\zeta_f(t) = \exp\left(\sum_{n=1}^{\infty} \frac{p_n(f)}{n} t^n\right).$$

Expanding out the powers of the series gives,

$$\zeta_f(t) = 1 + p_1(f).t + \frac{1}{2}[p_2(f) + p_1(f)^2].t^2 + \frac{1}{6}[2p_3(f) + 3p_2(f)p_1(f) + p_1(f)^3].t^3 + ...$$

For example, consider the dynamical system (Σ_2, σ). The full 2- shift Σ_2 is described by the transition matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Let us denote the eigen values 0 and 2 of the transition matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ by λ and μ respectively. Then, by Proposition 2.10, we have,

$$p_n(\sigma_A) = tr(A^n) = \lambda^n + \mu^n = 0^n + 2^n = 2^n$$

$$\therefore \quad \zeta_{\sigma_A}(t) = \exp\left(\sum_{n=1}^{\infty} \frac{p_n(\sigma_A)}{n} t^n\right) = \exp\left(\sum_{n=1}^{\infty} \frac{2^n}{n} t^n\right) = \exp\left(\sum_{n=1}^{\infty} \frac{(2t)^n}{n}\right)$$
$$= \exp\left(-\log(1-2t)\right)$$
$$= \frac{1}{1-2t}$$

The important key to derive the zeta function of the shift map of any finite type shift is:

Theorem: 4.1[1]: *If A is a $r \times r$ nonnegative integer matrix, $\chi_A(t)$ its characteristic polynomial and σ_A its associated shift map, then*

$$\zeta_{\sigma_A}(t) = \frac{1}{t^r \chi_A(t^{-1})} = \frac{1}{\left|\mathrm{I}_r - tA\right|} = \frac{1}{\prod_{\lambda \in sp^{\times}(A)} (1-\lambda t)}$$ *, where $sp^{\times}(A)$ is the nonzero spectrum of A.*

4.2: **Derivations of zeta function for the shift map σ on the full *m*-shift Σ_m :**

We know that the full *m*-shift Σ_m is described by the non-negative integer matrix *A* given by:

$$A = \begin{bmatrix} 1 & 1 & 1 & . & 1 \\ 1 & 1 & 1 & . & 1 \\ 1 & 1 & 1 & . & 1 \\ . & . & . & . & . \\ 1 & 1 & 1 & . & 1 \end{bmatrix}_{m \times m}$$

Here to find the zeta function of the shift map σ on the full *m*-shift Σ_m by fruitfully using the theorem 4.1, we need to compute $\left|\mathrm{I}_m - tA\right|$. We perform this as follows:

$$D = \left|\mathrm{I}_m - tA\right| = \begin{vmatrix} 1-t & -t & -t & -t & .. & -t \\ -t & 1-t & -t & -t & .. & -t \\ -t & -t & 1-t & -t & .. & -t \\ -t & -t & -t & 1-t & .. & -t \\ .. & .. & .. & .. & .. & .. \\ -t & -t & -t & -t & .. & 1-t \end{vmatrix}_{m \times m}$$

$$= \begin{vmatrix} 1-mt & -t & -t & -t & .. & -t \\ 1-mt & 1-t & -t & -t & .. & -t \\ 1-mt & -t & 1-t & -t & .. & -t \\ 1-mt & -t & -t & 1-t & .. & -t \\ .. & .. & .. & .. & .. & .. \\ 1-mt & -t & -t & -t & .. & 1-t \end{vmatrix}_{m \times m} \qquad [C_1^{'} = C_1 + C_2 + C_3 + ... + C_m]$$

$$= \begin{vmatrix} 1-mt & -t & -t & -t & .. & -t \\ 0 & 1 & 0 & 0 & .. & 0 \\ 0 & 0 & 1 & 0 & .. & 0 \\ 0 & 0 & 0 & 1 & .. & 0 \\ .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & .. & 1 \end{vmatrix}_{m \times m}$$

$$= (1-mt).1 = 1-mt$$

$$\therefore \zeta_{\sigma_m}(t) = \frac{1}{t^r \chi_A(t^{-1})} = \frac{1}{|\mathrm{I}_m - tA|} = \frac{1}{1-mt} \quad ■$$

5. ***Entropy of the full m-shift***:

Entropy is a very important and deeper concept in dynamics that measures the dynamical complexity of mappings. Topological entropy is a positive number assigned to every topological dynamical system that roughly tells us how much chaotic a dynamical system is. It generally gives the exponential rate of growth of the number of orbits distinguishable with finite but arbitrary precision. Metric entropy is closely related to topological entropy which not only measures the dynamical complexity of mappings, but also plays a very important role in the study of information theory. For shifts, entropy measures the *information capacity* or *transmissibility of messages*. The entropy of a shift is an important number invariant under conjugacy and behaves well under standard operations like factor codes and products. For a shift space *X*, the entropy of *X* is denoted as *h*(*X*) and defined as

$$h(X) = \lim_{n \to \infty} \frac{1}{n} \log |B_n(X)|$$

Though the concepts of *Perron-Frobenius theorem*, *Perron eigenvalue*, *Perron eigenvector* etc. are needed for rigorous calculation of entropy of topological Markov shifts, we need nothing other than the definition for the calculation of entropy for the full *m*-shift.

For the full *m*-shift $X = X_{[m]} = \Sigma_m$, $|B_n(X)| = m^n$. So, by simple calculation we have,

$$h(X) = \lim_{n \to \infty} \frac{1}{n} \log |B_n(X)| = \lim_{n \to \infty} \frac{1}{n} \log m^n = \lim_{n \to \infty} \frac{1}{n} .n \log m = \log m > 0$$

Thus the entropy for the full *m*-shift is simply $\log m > 0$ which indicates the dynamical complexity of the phase space Σ_m of the topological dynamical system (Σ_m, σ). ■

Conclusions:

In this paper we have mainly established that the shift transformation σ on the full *m*-shift Σ_m is *Devaney Chaotic*. To do this we have employed the concepts of graphs, matrix and linear algebra, topological Markov chains and metric spaces. In theorem 3.4, the well-known chaotic shift transformation σ on Σ_m have been shown to be generically δ-chaotic with $\delta = diam(\Sigma_m) = 1$. In theorem 3.6 and 3.7, we have proved that σ has respectively *modified weakly chaotic dependence* and *weakly chaotic dependence on initial conditions*. In the proof of both the theorems, Li-Yorke pairs have been very purposefully constructed. Further, the zeta function of this transformation has also been derived. Simple calculation of entropy for the full *m*-shift is given as routine work. The ways of establishing some results may be fruitfully employed for the same purpose in other topological Markov chains. Most of the results are quite interesting and might have profound applications in advanced analysis, theory of coding, representation of general dynamical systems and discrete mathematics.

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